JOURNAL OF APPROXIMATION THEORY 77, 153–166 (1994)

The Problem of A. F. Timan on the Precise Order of Decrease of the Best Approximations

R. K. S. RATHORE

Department of Mathematics, Indian Institute of Technology, Kanpur, Kanpur 208016, India

Communicated by Paul L. Butzer

Received February 19, 1992; accepted in revised form September 23, 1993

The problem of Timan on finding a necessary and sufficient condition for $A_\sigma(f)_{L_q} \sim \omega_k(f; 1/\sigma)_{L_q}$, $\sigma \to \infty$, is solved. The condition is $\omega_k(f; \delta)_{L_q} \sim \omega_{k+1}(f; \delta)_{L_q}$, $\delta \to 0$. Related problems in other situations have also been studied.

1. INTRODUCTION

Let $1 \leqslant q \leqslant \infty$, and $A_\sigma(f)_{L_q}$ denote the error in the best approximation of $f \in L_q$ on $(-\infty, \infty)$ by integral functions of finite degree $\leqslant \sigma$. For any positive integer k we have the well-known Jacksonian estimate

$$A_\sigma(f)_{L_q} \leqslant C_k \omega_k(f; 1/\sigma)_{L_q}, \tag{1.1}$$

in which the constant C_k does not depend on σ or f (A. F. Timan [18, p. 262]), due to N. I. Akhiezer [1] ($k=1$, $q<\infty$, and $k=2$) and A. F. Timan and M. F. Timan [17] ($k \geqslant 3$). The corresponding inequality in the trigonometric case for continuous functions is due to D. Jackson [10] ($k=1$), N. I. Akhiezer [1] ($k=2$), and S. B. Stechkin [15] ($k \geqslant 3$).

The problem of A. F. Timan concerns conditions on f with which the reverse inequality holds

$$\omega_k(f; 1/\sigma)_{L_q} \leqslant E A_\sigma(f)_{L_q}, \tag{1.2}$$

with E independent of σ. If this happens both $A_\sigma(f)$ and $\omega_k(f; 1/\sigma)$ have precisely the same rate of decrease to zero as $\sigma \to \infty$.

There have been many notable contributors to the problem by way of sufficient conditions in various special cases (see A. F. Timan [18], especially Chap. 6 and 7, for a historical background) while the culmination of these efforts occurs in the studies of the general case by A. F. Timan

0021-9045/94 $6.00

himself, resulting in the following sufficient, but not necessary, condition (A. F. Timan [18, p. 411]) for (1.2) to hold:

$$t^k \int_t^1 (\omega_k(f; u)_{L_q}/u^{k+1})\, du = O(\omega_k(f; t)_{L_q}), \qquad t \to 0. \tag{1.3}$$

This incidentally is the last statement of the monograph concerning the equivalence of the degree of best approximation and the moduli of smoothness of the functions. This relation (1.3), however, is necessary and sufficient for

$$(1/n^k) \sum_{v=0}^{n} (v+1)^{k-1} A_v(f)_{L_q} = O(A_n(f)_{L_q}), \qquad n \to \infty, \tag{1.4}$$

to hold. In fact, (1.2) follows from the equivalence of (1.3) and (1.4) in view of the gereral estimate

$$\omega_k(f; 1/n)_{L_q} \leqslant (B_k/n^k) \sum_{v=0}^{n} (v+1)^{k-1} A_v(f)_{L_q}, \qquad n \to \infty, \tag{1.5}$$

(A. F. Timan [18, p. 342]) in which B_k is independent of f and n. The corresponding inequality in the trigonometric case is due to A. F. Timan and M. F. Timan [16], ($q < \infty$) and the extension to the case $q = \infty$, subject to the continuity of f is by S. B. Stechkin [15]. The inequalities (1.3) and (1.4) are also equivalent in the trigonometric case (A.F. Timan [18, p. 405]), in which for $q = \infty$, in the case when $f(x)$ is continuous, an equivalent statement is also contained in S. M. Lozinskii [11] and N. K. Bari and S. B. Stechkin [2].

A related problem for $C_{2\pi}$, the space of 2π-periodic continuous functions, was posed by S. B. Stechkin in 1977 in the form of the question "if for $f \in C_{2\pi}$ which does not belong to $C_{2\pi}^{\infty}$, there exists a k such that $\omega_k(f; 1/n) \leqslant CE_n^*(f)$?" for which a counter-example was given by J. Boman [6]. (As very kindly pointed out by a referee, the question is indeed connected with an old problem of J. Favard; see R. J. Nessel and E. van Wickeren [13], W. Dickmeis, R. J. Nessel, and E. van Wickeren [8], and W. Dickmeis and R. J. Nessel [7] for a general approach via quantitative comparison versions of the uniform boundedness principle.)

More recently, Z. Ditzian [9] generalized the methodology of the above works by providing an axiomatic set-up for similar sufficient conditions for the equivalence of rate of approximation and smoothness in many more situations.

In the sequel "$a(\sigma) \sim b(\sigma)$, $\sigma \to \infty$" signifies both the relations $a(\sigma) = O(b(\sigma))$ and $b(\sigma) = O(a(\sigma))$ as $\sigma \to \infty$, or, equivalently, that the lower and the upper limits $\liminf_{\sigma \to \infty} a(\sigma)/b(\sigma)$ and $\limsup_{\sigma \to \infty} a(\sigma)/b(\sigma)$ exist and are positive. If for a class C of pairs $(a(\sigma), b(\sigma))$ the set of the lower limits is bounded away from zero and the set of upper limits is bounded, we say that the asymptotic equivalence holds uniformly in C. Writing just "$a(\sigma) \sim b(\sigma)$" would mean that there exist positive constants α and β such that $\alpha \leqslant a(\sigma)/b(\sigma) \leqslant \beta$ for the entire range of σ under consideration (all $\sigma \in (0, \infty)$ in the present case) and, moreover, the uniformity of $a(\sigma) \sim b(\sigma)$ for (a, b) in a certain class C refers to the set of such α's being bounded away from zero and that of the β's being bounded. The usages "$a(\delta) \sim b(\delta)$, $\delta \to 0$" and "$a(n) \sim b(n)$, $n \to \infty$" (with n's taking positive integral values), etc. are in a similar sense.

Even though an implicit concern with the precise order of decrease of the best approximations has been there for a long time (see e.g., S. N. Bernstein [4–5]), a probable reason why the problem of finding the necessary and sufficient conditions remained unsettled until now is that it is not possible to characterize all the functions f satisfying

$$A_\sigma(f)_{L_q} \sim \omega_k(f; 1/\sigma)_{L_q}, \quad \sigma \to \infty, \tag{1.6}$$

i.e., functions f for which the best approximations $A_\sigma(f)_{L_q}$ are precisely of the same order as $\omega_k(f; 1/\sigma)_{L_q}$, in terms of $\omega_k(f; \delta)_{L_q}$ alone. The same holds in many other situations including the trigonometric approximation in $C_{2\pi}$.

The reason should not be surprising in view of the classical Bernstein and Zygmund theorems which in the $C_{2\pi}$ case enable us to state that $E_n^*(f) \sim (1/n)^\alpha$ is equivalent to $\omega(f; \delta) \sim \delta^\alpha$, provided that $0 < \alpha < 1$, while for $\alpha = 1$, the corresponding condition $\omega_2(f; \delta) \sim \delta$ is not in terms of $\omega(f; \delta)$ and involves $\omega_2(f; \delta)$. Thus $E_n^*(|\sin x|) \sim \omega(|\sin x|; 1/n)$ actually holds not just because $\omega(|\sin x|; \delta) \sim \delta$, but because also $\omega_2(|\sin x|; \delta) \sim \delta$. (Note that $\omega(\sin x; \delta) \equiv \omega(|\sin x|; \delta)$!)

In this paper we solve the above problem of A. F. Timan for A_σ, of S. B. Stechkin for E_n^*, and similar other problems, e.g., in the best approximation of functions $f \in L_p[-1, 1]$ by algebraic polynomials, arising in Z. Ditzian [9]. In a sequel to this paper we provide an axiomatic set-up for such equivalence as well as inverse theorems of a general order both in approximation by sequences of linear operators and best approximations from an increasing sequence of subspaces of a Banach space, possessing Jackson and Bernstein type inequalities, as in Z. Ditzian [9] and M. Becker and R. J. Nessel [3]. Let us begin with:

2. A Consequence of the Lethargy Theorem of S. N. Berstein

Stated for $C_{2\pi}$, the lethargy theorem (S. N. Bernstein [5]) says that if $\{E_n\}$ is a non-increasing sequence of non-negative numbers converging to 0, there exists an $f \in C_{2\pi}$ for which the error $E_n^*(f)$ in the best approximation by trigonometric polynomials of order n equals E_n for all n. Hence, let α be a positive number and let $f \in C_{2\pi}$ correspond to

$$E_n = \begin{cases} 1, & n=1 \\ (k!)^{-\alpha}, & (k-1)!+1 \leqslant n \leqslant k!,\ k>1. \end{cases}$$

We have the following observations on this function:

(1) If $\alpha < 1$, $f \in \operatorname{Lip} \alpha$, but for no $\delta > 0$ does f belong to $\operatorname{Lip}(\alpha+\delta)$. If $\alpha = m$, an integer, $f^{(m-1)}$ belongs to Zygmund class but $f^{(m)}$ does not exist continuously. If $\alpha = m+\delta$, where m is a positive integer and $0<\delta<1$, $f^{(m)}$ exists and belongs to $\operatorname{Lip} \delta$, but does not belong to $\operatorname{Lip}(\delta')$ for any $\delta' > \delta$, so that f is not $m+1$ times continuously differentiable. Thus, for no α does f belong to $C_{2\pi}^{\infty}$.

(2) $E_n^*(f) = O(n^{-\alpha})$, but $E_n^*(f) \neq o(n^{-\alpha})$.

(3) $E_n^*(f) \sim \omega_k(f; 1/n)$ does not hold for any $k \geqslant 1$.

The assertions (1) and (2) are clear in view of the Bernstein and Zygmund theorems. For (3), if there exist positive constants β and γ such that

$$\gamma \omega_k(f; 1/n) \leqslant E_n^*(f) \leqslant \beta \omega_k(f; 1/n), \qquad n = 1, 2, ...,$$

then

$$\frac{E_n}{E_{2n}} \leqslant \frac{\beta \omega_k(f; 1/n)}{\gamma \omega_k(f; 1/2n)} \leqslant \frac{2^k \beta}{\gamma} < \infty, \qquad n = 1, 2, ...,$$

which does not agree with the fact that

$$E_{m!}/E_{2(m!)} = \left\{\frac{m!}{(m+1)!}\right\}^{-\alpha} = (m+1)^{\alpha} \to \infty, \qquad m \to \infty.$$

Thus just this one function f, constructed above, provides a counter-example for both (cf. Ditzian [9, p. 317], Boman [6]) the assertion "$0<\alpha<k$, $\omega_k(f; t) = O(t^{\alpha})$ and $\omega_k(f; t) \neq o(t^{\alpha}) \Rightarrow \omega_k(f, 1/n) \sim E_n^*(f)$" and the problem of S. B. Stechkin: "if for $f \in C(T)$ which does not belong to $C^{\infty}(T)$ there exists an r such that $\omega_r(f, 1/n) \leqslant C E_n^*(f)$?". Here $C(T)$ and $C^{\infty}(T)$ are the same as $C_{2\pi}$ and $C_{2\pi}^{\infty}$, respectively.

Since the Bernstein lethargy theorem is valid for best approximations in a general Banach space X from an increasing sequence of finite dimensional subspaces $\{X_n\}$, the union of which is dense in X, it clearly provides us with a powerful methodology for obtaining similar counter examples in similar other contexts (e.g., trigonometric and algebraic polynomial approximation in L_p-spaces), too.

3. Solution of the Problems of A. F. Timan and S. B. Stechkin

The solution of the problems, interestingly enough, is Bernsteinian and Zygmundian both, in the sense that ω_k and ω_{k+1} together are required to characterize (1.6) and similar equivalences in the trigonometric, etc., cases. To relate (1.6) with the following result we note that for any one particular function f, the asymptotic equivalence (1.6) is equivalent to the equivalence $A_\sigma(f)_{L_q} \sim \omega_k(f; 1/\sigma)_{L_q}$.

Theorem 3.1. *Let m be a positive integer and $f \in L_q(-\infty, \infty)$. There exists a positive constant G such that*

$$\omega_m(f; 1/\sigma)_{L_q} \leqslant G A_\sigma(f)_{L_q}, \qquad \sigma > 0, \tag{3.1}$$

if, and only if, there exists a positive constant F such that

$$\omega_m(f; \delta)_{L_q} \leqslant F \omega_{m+1}(f; \delta)_{L_q}, \qquad \delta > 0. \tag{3.2}$$

Proof. If (3.1) holds, using (1.1) for $k = m+1$, (3.2) follows with $F = GC_{m+1}$. Conversely, if (3.2) holds, using the well-known inequality

$$\omega_m(f; u)_{L_q} \leqslant (1 + u/t)^m \, \omega_m(f; t)_{L_q}, \qquad (t < u),$$

we get

$$\begin{aligned}
&t^{m+1} \int_t^1 [\omega_{m+1}(f; u)_{L_q}/u^{m+2}] \, du \\
&\leqslant 2t^{m+1} \int_t^1 [\omega_m(f; u)_{L_q}/u^{m+2}] \, du \\
&\leqslant 2t^{m+1} \omega_m(f; t)_{L_q} \int_t^1 [(1 + u/t)^m / u^{m+2}] \, du \\
&\leqslant 2^{m+1} \omega_m(f; t)_{L_q} \\
&\leqslant 2^{m+1} F \omega_{m+1}(f; t)_{L_q}.
\end{aligned}$$

Thus (1.3) holds with $k = m+1$, so that for some constant E independent of σ, we have

$$\omega_m(f; 1/\sigma)_{L_q} \leqslant F \omega_{m+1}(f; 1/\sigma)_{L_q} \leqslant FEA_\sigma(f)_{L_q},$$

and (3.1) follows with $G = FE$, completing the proof.

Note that Theorem 3.1 asserts that (1.3) for $k=m$ necessarily implies (3.2). Examples where (3.2) holds, but (1.3) fails for $k=m$, are presented in Section 5. Since for any C^∞-function f of compact support (or, more generally, such that $f^{(m)} \in L_q(-\infty, \infty)$, $m \geq 1$) there holds

$$\omega_m(f; \delta)_{L_q} \sim \delta^m, \qquad m = 1, 2, ...,$$

such a function trivially fails to satisfy (3.2) for any m. An example of a non-C^∞ function for which (3.2) fails for every m will be given in Section 5, where it will also be seen that, unlike the C_k in (1.1), the constants G and F in Theorem 3.1 do depend on the function f under consideration.

COROLLARY 3.2. *If $k < m$, for the following statements there holds* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v):

(i) $\omega_k(f; 1/\sigma)_{L_q} \sim A_\sigma(f)_{L_q}$, $\sigma \to \infty$;

(ii) $\omega_k(f; \delta)_{L_q} \sim \omega_m(f; \delta)_{L_q}$, $\delta \to 0$;

(iii) $t^m \int_t^1 (\omega_m(f; u)_{L_q}/u^{m+1})\, du = O(\omega_m(f; t)_{L_q})$, $t \to 0$;

(iv) $(1/n^m) \sum_{\nu=0}^{n} (\nu+1)^{m-1} A_\nu(f)_{L_q} = O(A_n(f)_{L_q})$, $n \to \infty$;

(v) $\omega_m(f; 1/\sigma)_{L_q} \sim A_\sigma(f)_{L_q}$, $\sigma \to \infty$.

We note that the above results and the proof remain valid in the $C_{2\pi}$ as well as the L_q-case of approximation of 2π-periodic functions by trigonometric polynomials, where A_σ and σ are replaced by E_n^* and n, respectively, giving rise to the following theorem, solving the problem of S. B. Stechkin:

THEOREM 3.3. *Let m be a positive integer, f a 2π-periodic function belonging to L_q over the period and $E_n^*(f)_{L_q}$ denote the error in the L_q-best approximation of f by trigonometric polynomials of order n. There exists a positive constant G such that*

$$\omega_m(f; 1/n)_{L_q} \leq G E_n^*(f)_{L_q}, \qquad \sigma > 0, \tag{3.3}$$

if, and only if, there exists a positive constant F such that

$$\omega_m(f; \delta)_{L_q} \leq F \omega_{m+1}(f; \delta)_{L_q}, \qquad \delta > 0. \tag{3.4}$$

In a similar fashion, using Thm. 5.1 of Z. Ditzian [9] we obtain the following generalization for a Banach space X of functions or distributions on the unit circle $\mathbb{T}$ (equivalently 2π-periodic case on the real line). For X the shift operator S_h is defined by $S_h f(x) = f(x+h)$, $x, h \in T$, for functions and dually by $\langle s_h f, g \rangle = \langle f, s_{-h} g \rangle$, for $f \in \mathcal{S}'$, the space of tempered distributions dual to the space $\mathcal{S}$ of test functions:

THEOREM 3.4. *If X is Banach space of functions or distributions on unit circle $\mathbb{T}$ for which a shift operator S_h is a weakly or weakly* continuous isometry, then*

$$E_n(f)_X \sim \omega^r(f, 1/n)_X \qquad \text{iff} \quad \omega^r(f, t)_X \sim \omega^{r+1}(f, t)_X.$$

A multivariate generalization of Theorem 3.4, (using Thm. 9.4 of Z. Ditzian [9]) for a Banach space X of functions or distributions on a torus $\mathbb{T}^d$, with the best approximations

$$E_n(f)_X = \inf_{t \in \mathcal{T}_n} \|f - \tau\|_X,$$

where $\mathcal{T}_n$ is the set of trigonometric polynomials of degree n in each direction and the modulus of smoothness is given by

$$\omega^r(f, t)_X = \sup_{0 < |v| \leqslant t} \|\Delta_v^r f\|_X,$$

where Δ_v^r is the r-times iterated $\Delta_v = S_v - I$, S_v denotes a shift by v, and I is the identity operator, is as follows:

THEOREM 3.5. *If X is a Banach space of functions or distributions on $\mathbb{T}^d$ in which a shift operator is weakly or weakly* continuous isometry, then $E_n(f)_X \sim \omega^r(f, 1/n)_X$ iff $\omega^r(f, t)_X \sim \omega^{r+1}(f, t)_X$.*

Results analogous to that of Corollary 3.2 are valid also for Theorems 3.3–3.5 (and also others occurring in the sequel).

4. THE BEST APPROXIMATION BY ALGEBRAIC POLYNOMIALS

Following Z. Ditzian [9], let us consider $f \in L_p[-1, 1]$, $(1 \leqslant p \leqslant \infty)$ and the error in its best $L_p[-1, 1]$ approximation

$$E_n(f)_p = \inf_{g \in \mathcal{P}_n} \|g - f\|_p, \tag{4.1}$$

where $\mathcal{P}_n$ denotes the space of all algebraic polynomials of degree $\leqslant n$. In this situation the modulus of smoothness

$$\omega_\varphi^r(f, t)_p = \sup_{0 < h \leqslant t} \|\delta_{h\varphi}^r f\|_p, \qquad (\varphi(x) = \sqrt{(1 - x^2)}), \tag{4.2}$$

where δ_η stands for the central difference operator of step size η, with the understanding that $\delta_\eta^r f(x)$ is zero if the interval $(x-\eta r/2, x+\eta r/2)$ is not contained in $[-1, 1]$, is equivalent to the K-functional

$$K_{r,\varphi}(f, t^r)_m = \inf_{g^{(r-i)} \in AC_{\mathrm{loc}}} (\|f-g\|_p + t^r \|\varphi^r g^{(r)}\|_p). \tag{4.3}$$

and there hold the Jackson inequality

$$E_n(f)_p \leqslant C\omega_\varphi^r(f, 1/n)_p, \tag{4.4}$$

(see [9, (6.3), p. 330]) and the Bernstein inequality (see [9, (6.4), p. 331])

$$\|\varphi^r p^{(r)}\|_p \leqslant Cn^r \|p\|_p, \qquad p \in \mathscr{P}_n,\ 1 \leqslant p \leqslant \infty, \tag{4.5}$$

which Z. Ditzian [9, Thm. 6.1] used to prove that the conditions $\omega_\varphi^r(f, t)_p \sim \psi(t^r)$ and $E_n(f)_p \sim \psi(1/n^r)$ are equivalent and thus either implies $E_n(f)_p \sim \omega_\varphi^r(f, 1/n)_p$, provided $\psi \in S$ (S for steady), which by definition means that ψ is a continuous and non-decreasing function on $[0, 1]$ such that $0 = \psi(0) < \psi(t)$ for $t > 0$ and

$$\delta \int_\delta^1 \frac{\psi(t)}{t^2}\, dt \sim \psi(\delta). \tag{4.6}$$

Let us observe that there exist constants L, M such that

$$\omega_\varphi^{r+1}(f, t)_p \leqslant L\omega_\varphi^r(f, t)_p \tag{4.7}$$

and that, as in the case of non-weighted moduli,

$$\omega_\varphi^r(f, \lambda t)_p \leqslant M(1+\lambda)^r\, \omega_\varphi^r(f, t)_p. \tag{4.8}$$

Hence, if $\omega_\varphi^m(f, t)_p \leqslant F\omega_\varphi^{m+1}(f, t)_p$ for some constant F, using (4.7) and (4.8) with $r = m$, we have

$$\begin{aligned}
&t \int_t^1 [\omega_\varphi^{m+1}(f, u^{1/(m+1)})_p/u^2]\, du \\
&\leqslant Lt \int_t^1 [\omega_\varphi^m(f, u^{1/(m+1)})_p/u^2]\, du \\
&\leqslant Lt\omega_\varphi^m(f, t^{1/(m+1)})_p \int_t^1 M[(1+(u/t)^{1/(m+1)})^m/u^2]\, du \\
&\leqslant 2^m LM\omega_\varphi^m(f, t^{1/(m+1)})_p \int_t^1 [(t/u)^{1/(m+1)}]/u\, du \\
&\leqslant (m+1)\, 2^m LMF\omega_\varphi^{m+1}(f, t^{1/(m+1)})_p,
\end{aligned}$$

showing that $\psi(t)=\omega_\varphi^{m+1}(f, t^{1/(m+1)})_p \in S$. Hence $E_n(f) \sim \omega_\varphi^{m+1}(f, 1/n)_p$. The converse follows from (4.4) (with $r=m+1$) and (4.7) (with $r=m$). As a consequence, we obtain:

THEOREM 4.1. $E_n(f)_p \sim \omega_\varphi^m(f, 1/n)_p$ iff $\omega_\varphi^m(f, t)_p \sim \omega_\varphi^{m+1}(f, t)_p$.

This result extends to multivariate best algebraic polynomial approximation as follows. Let P_n denote the set of polynomials, in d variables, of total degree n and

$$E_n(f)_{L_p(\mathbb{S}^d)} = \inf\{\|f-P\| : P \in P_n\},$$

where the domain $\mathbb{S}^d \subset \mathbb{R}^d$ is a simple polytope (each vertex having exactly d edges) with a non-empty interior. In this case the measures of smoothness $\omega_{\mathbb{S}}^r(f, t)_p$ and $\tilde{\omega}_{\mathbb{S}}^r(f, t)_p$ are defined according to Remark 9.2 in Z. Ditzian [9], where it is shown [9, p. 334, Th. 9.1] that for $\psi \in S$ the statements

$$E_n(f)_{L_p(\mathbb{S})} \sim \psi(1/n^r), \tag{4.9}$$

$$\omega_{\mathbb{S}}^r(f, t)_p \sim \psi(t^r), \tag{4.10}$$

$$\tilde{\omega}_{\mathbb{S}}^r(f, t)_p \sim \psi(t^r), \tag{4.11}$$

are equivalent. Following the proof of Theorem 4.1 and using the analogs of (4.7–4.8) for these moduli we thus have:

THEOREM 4.2. *Let $\mathbb{S}$ be a simple polytype, $\mathbb{S} \subset \mathbb{R}^d$, $1 \leqslant p \leqslant \infty$, and let $\Omega_{\mathbb{S}}^r(f, t)_p$ stand for any one of $\tilde{\omega}_{\mathbb{S}}^r(f, t)_p$ or $\omega_{\mathbb{S}}^r(f, t)_p$. Then*

$$E_n(f)_{L_p(\mathbb{S})} \sim \Omega_{\mathbb{S}}^m(f, 1/n)_p \qquad \textit{iff} \quad \Omega_{\mathbb{S}}^m(f, t)_p \sim \Omega_{\mathbb{S}}^{m+1}(f, t)_p.$$

5. CONCLUDING REMARKS AND EXAMPLES

To summarize, in the classical cases of best approximation of functions by integral functions of finite degree, trigonometric polynomials, and algebraic polynomials, we have shown that the equivalences (such as (3.2) and (3.4)) of a modulus of smoothness with that of higher order provide the necessary and sufficient conditions for the corresponding Jackson type estimates to be precise.

The general mechanism, using Bernstein's lethargy theorem, allows us to construct examples of non-C^∞ functions for which the Jackson type estimates in the L_q-best approximation ($1 \leqslant q \leqslant \infty$) by trigonometric as well as algebraic polynomials cannot be precise. In the $L_q(-\infty, \infty)$

approximation by integral functions, however, the subspaces involved are not finite dimensional so that, as such, the method of Section 2 is not applicable. For $q=2$, nevertheless, such constructions are possible in view of the Paley–Wiener theorem and a theorem due to I. S. Tyuremskih (Thm. 6.7, p. 151, I. Singer [14]) on best approximations from an increasing sequence of closed subspaces of a Hilbert space. Moreover, for $q=\infty$, such functions in the trigonometric approximation in $L_\infty(-\pi, \pi]$ serve equally well as the required examples for $L_\infty(-\infty, \infty)$ (in view of Theorems 3.1–3.2). Can we find such functions also for the remaining $q \in [1, \infty]$? Moreover, do there exist non-trivial functions for which conditions such as (3.2) hold while those of type (1.3) do not? The constructions below give an affirmative answer to both these questions:

EXAMPLE 5.1. Let $f \in L_q(-\pi, \pi] \backslash C^\infty$ be constructed by the method of Section 2 such that (3.3) holds for no m. By Theorem 3.3, (3.4) holds for no m. Let $g(x)$ be a C^∞-function vanishing outside $[-3\pi, 3\pi]$ such that $g(x)=1$ for $x \in [-2\pi, 2\pi]$. Defining $F(x)=f(x)\,g(x)$, $x \in (-\infty, \infty)$, it is obvious that for all $t>0$,

$$\omega_m(F; t)_{L_q(-\infty,\infty)} \geqslant \omega_m(f; t)_{L_q(-\pi,\pi]}.$$

For a reverse inequality, using the discrete Leibniz rule

$$\Delta_h^m F(x) = \sum_{k=0}^{m} \binom{m}{k} \Delta_h^k f(x)\, \Delta_h^{m-k} g(x+kh),$$

it follows that

$$\omega_m(F; t)_{L_q(-\infty,\infty)} = O\left(\sum_{k=1}^{m} t^{m-k} \omega_k(f; t)_{L_q(-\pi,\pi]} + t^m\right), \qquad t \to 0.$$

Then, using (A. F. Timan [18, 3.3.(12) and 3.3.4, pp. 110–112])

$$\omega_k(f; t)_{L_q(-\pi,\pi]} = O\left(t^k \int_t^c u^{-(k+1)} \omega_m(f; u)_{L_q(-\pi,\pi]}\, du + t^k\right),$$

where c is a positive constant and $1 \leqslant k < m$, we have

$$\begin{aligned} t^{m-k}&\omega_k(f; t)_{L_q(-\pi,\pi]} \\ &= O\left(\omega_m(f; t)_{L_q(-\pi,\pi]} \int_t^c (t+u)^m u^{-(k+1)}\, du + t^m\right) \\ &= O(\omega_m(f; t)_{L_q(-\pi,\pi]} + t^m), \qquad t \to 0. \end{aligned}$$

Hence, in view of (A. F. Timan [18, 3.3(7), p. 107])

$$t^m = O(\omega_m(f; t)_{L_q(-\pi,\pi]}), \qquad t \to 0,$$

there follows the required reverse inequality

$$\omega_m(F; t)_{L_q(-\infty,\infty)} \leqslant A_m \omega_m(f; t)_{L_q(-\pi,\pi]}, \qquad t > 0,$$

where the constant A_m does not depend on t. Thus there holds

$$\omega_m(F; \delta)_{L_q(-\infty,\infty)} \sim \omega_m(f; \delta)_{L_q(-\pi,\pi]},$$

for all orders $m \geqslant 1$. Hence, as $f(x)$ does not satisfy (3.4), the function $F(x)$ cannot satisfy (3.2) and by Theorem 3.1, (3.1) for this function cannot hold for any $m \geqslant 1$. Note that $F(x)$ is not a C^∞-function and that in showing that it does not satisfy (3.1) our Theorems 3.1–3.2 have been crucially used.

EXAMPLE 5.2. An illustration of the applicability of our results in all the three types of approximations that we have considered is as follows. With $\beta \geqslant 0$, and $1 \leqslant q \leqslant \infty$, we define

$$F_\beta(x) = (1-2x^2)_+^\beta = \begin{cases} (1-2x^2)^\beta, & |x| \leqslant 1/\sqrt{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f_\beta^1 \in L_q(-\infty, \infty)$ equal F_β, $f_\beta^2 \in L_q(-\pi, \pi]$ denote the 2π-periodic extension of the restriction of F_β to $(-\pi, \pi]$, and $f_\beta^3 \in L_q[-1, 1]$ be the restriction of F_β to the interval $[-1, 1]$, respectively. Let for $m = 1, 2, ..., \omega_m(f_\beta^*; \delta)_q$ commonly denote the moduli of smoothness of mth order of f_β^1, f_β^2, and f_β^3 over the respective domains. Simple computations show that for all $\beta \geqslant 0$, $1 \leqslant q \leqslant \infty$, and $m = 1, 2, ...,$ there holds

$$\omega_m(f_\beta^*; \delta)_q \sim \begin{cases} \min(1, \delta^{\beta+1/q}), & \beta + 1/q \leqslant m, \\ \min(1, \delta^m), & \beta + 1/q > m. \end{cases}$$

It follows that there exists a positive number F such that

$$\omega_m(f_\beta^*; \delta)_q \leqslant F\omega_{m+1}(f_\beta^*; \delta)_q,$$

iff $m \geqslant \beta + 1/q$. (Note that if $\beta + 1/q = m$, (1.3) does not hold for $k = m$.) Accordingly, using Theorem 3.1, we have

$$A_\sigma(f_\beta^1)_{L_q} \sim \omega_m(f_\beta^*; 1/\sigma)_q, \qquad \text{iff} \quad m \geqslant \beta + 1/q,$$

and hence that

$$A_\sigma(f_\beta^1)_{L_q} \sim \min(1, \sigma^{-(\beta+1/q)}).$$

For the functions f_β^2 and f_β^3, exactly similar conclusions follow from Theorem 3.3 and Theorem 4.1, respectively.

EXAMPLE 5.3. Let us also draw a conclusion of a general interest using some more examples where the conditions of type (3.2) are applicable while those of the type (1.3) fail. For this, let C denote the class of functions $f \in C_{2\pi}$ that satisfy the equivalence

$$E_n^*(f) \sim \omega(f; 1/n), \tag{5.1}$$

for which our "iff" condition (3.4) from Theorem 3.2 reads

$$\omega(f; t) \sim \omega_2(f; t), \tag{5.2}$$

and which we compare with the earlier "sufficient" condition

$$\delta \int_\delta^1 [\omega(f; t)/t^2]\, dt \sim \omega(f; \delta). \tag{5.3}$$

Let m be a positive integer and consider the function $g(x) = |\sin x| + \sin 10^m x \in C_{2\pi}$. For this function g, $\omega(g; t) \sim \omega_2(g; t)$ $(\sim t)$, $t \in (0, 1]$. Thus, (5.2) is applicable and hence (5.1) holds, i.e., $g \in C$. Note that (5.3) does not hold. Let us further observe that for $n \geqslant 10^m$, $E_n^*(g) \geqslant 1/(4\pi n)$ (I. P. Natanson [12, p. 170]) and $\lim_{n \to \infty} n\omega(g; 1/n) = 10^m + 1$. Hence, the estimate

$$c \leqslant \liminf_{n \to \infty} [\omega(g, 1/n)]^{-1} E_n^*(g)$$

cannot hold for any $c \geqslant 1/(4\pi + 10^m + 1)$. Since m is arbitrary, it follows that (5.1) *does not hold uniformly for all* $f \in C$. For the same reason the corresponding *asymptotic equivalence*

$$E_n^*(f) \sim \omega(f; 1/n), \qquad n \to \infty, \tag{5.4}$$

too, cannot hold uniformly for all $f \in C$. As $\lim_{\delta \to 0} \omega_2(g; \delta)/\delta = 2$, we conclude that *neither* (5.2), *nor its asymptotic version*

$$\omega(f; t) \sim \omega_2(f; t), \qquad t \to 0 \tag{5.5}$$

can hold uniformly for all $f \in C$.

Similar conclusions can be drawn for the spaces $L_q(-\infty, \infty)$, $L_q(-\pi, \pi)$ and $L_q[-1, 1]$, $1 \leqslant q \leqslant \infty$, by additively modifying (i.e., using analogs of the additive term $\sin 10^m x$) the functions f_β^1, f_β^2, and f_β^3 in Example 5.2, where $\beta = m - 1/q$. Thus, *for all m and q the constants G and F in* (3.1)–(3.2), (3.3)–(3.4), *etc., do depend on the particular function f under consideration, and cannot be replaced by universal constants.*

EXAMPLE 5.4. As the final example, we may cite the function

$$F(x) = |\sin x| \, (1 - \ln |\sin x|) \in C_{2\pi},$$

(with $F(m\pi) = 0$, when m is an integer), for which the order of approximation $E_n^*(f)$ is inferior to the Zygmundian order $1/n$, but is superior to all Bernsteinian orders $1/n^\alpha$, $0 < \alpha < 1$, and yet to which (5.2) is applicable since

$$\omega_2(F, \delta) = 2\omega(F, \delta) = \begin{cases} F(\delta), & \delta \in (0, \pi/2] \\ F(\pi/2), & \delta \in (\pi/2, \pi], \end{cases} \tag{5.6}$$

so that (5.1) holds. Note that $\omega(F; t) \cong t(\ln(1/t))$, $t \to 0$, whereas $\delta \int_\delta^1 [\omega(F; t)/t^2] \, dt \cong (\delta/2)(\ln(1/\delta))^2$, $\delta \to 0$, so that (5.3) fails. This example remains valid for $L_\infty(-\infty, \infty)$ (and also for $L_\infty[-1, 1]$, with "$\sin x$" replaced by "x"). Many more such orders of approximation for which the condition (5.3) does not hold but where our condition (5.2) is valid could be constructed by using the fact that any function $f \in C_{2\pi}$, which is concave in the interval $[0, \pi/2]$, is symmetric about 0 and $\pi/2$, and vanishes at the origin satisfies (5.6).

ACKNOWLEDGMENTS

The author records his deep gratitude to Prof. Dr. P. L. Butzer and a very kind referee for their constructive suggestions, including valuable advice on recent works, without which the present scope of the work could not have been possible.

REFERENCES

1. N. I. AKHIEZER, "Lectures on the Theory of Approximation," Gostekhizdat, 1974.
2. N. K. BARI AND S. B. STECHKIN, Best approximation and differential properties of two conjugate functions, *Trudy Moskov. Mat. Obshch.* **5** (1956), 483–522.
3. M. BECKER AND R. J. NESSEL, An elementary approach to inverse approximation theorems, *J. Approx. Theory* **23** (1978), 99–103.
4. S. N. BERNSTEIN, "The Extremal Properties of Polynomials and the Best Approximation of Continuous Functions of One Real Variable," M.-L., GONTI, 1937.
5. S. N. BERNSTEIN, On the inverse problem of the theory of the best approximation of continuous functions, *C. R. Acad. Sc.* (1938), "Collected Works," Vol. II, pp. 292–294, [Russian] Akad. Nauk SSSR, Moscow, 1954.
6. J. BOMAN, A problem of Stečkin on trigonometric approximation, *in* "Constructive Function Theory '77, Proceedings, Blagoevgrad 1977" (Bl. Sendov and D. Vacov, Eds.), pp. 269–273, Bulg. Acad. Sci., Sofia, 1980.
7. W. DICKMEIS AND R. J. NESSEL, Condensation principle with rates, *Studia Math.* **75** (1982), 55–68.
8. W. DICKMEIS, R. J. NESSEL, AND E. VAN WICKEREN, Quantitative extensions of the uniform boundedness principle, *Jahresber. Deutsch. Math.-Verein.* **89** (1987), 105–134.

9. Z. DITZIAN, Equivalence of rate of approximation and smoothness, *J. Approx. Theory* **62** (1990), 316–339.
10. D. JACKSON, "Über Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische summengegebener Ordnung," Diss., Göttingen, 1911.
11. S. M. LOZINSKII, The converse of Jackson's theorem, *Dokl. Akad. Nauk USSR* **83**, No. 5 (1952), 645–647.
12. I. P. NATANSON, "Constructive Function Theory," Vol. I., Ungar, New York, 1964.
13. R. J. NESSEL AND E. VAN WICKEREN, Negative results in connection with Favard's problem on the comparison of approximation processes, *in* "Anniversary Volume on Approximation Theory and Functional Analysis" (P. L. Butzer, R. L. Stens, and B. Sz. Nagy, Eds.), pp. 189–200, Birkhäuser-Verlag, Basel, 1984.
14. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer-Verlag, Berlin, 1970.
15. S. B. STECHKIN, On the order of the approximations of continuous functions, *Izv. Akad. Nauk USSR Ser. Mat.* **15** (1951), 219–242.
16. A. F. TIMAN AND M. F. TIMAN, The generalized modulus of continuity and the best mean approximation, *Dokl. Akad. Nauk USSR* **71**, No. 1 (1950), 17–20.
17. A. F. TIMAN AND M. F. TIMAN, On the relation between the moduli of smoothness of functions defined on the whole real axis, *Dokl. Akad. Nauk. USSR* **113**, No. 5 (1957), 995–997.
18. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," Hindustan (India), Delhi, 1966.